Math 564: Advance Analysis 1
Lecture 16

Example. For sets I, J (e.g. $I = \mathbb{N} = J$) and counting measures μ & ν on I & J, resp., we already know Fubini–Tonelli from basic analysis:

(a) Tonelli: For nonnegative $(a_{ij})_{i\in I, j\in J}$,

$$\sum_{j\in J}\sum_{i\in I} a_{ij} = \sum_{(i,j)\in I\times J} a_{ij} = \sum_{i\in I}\sum_{j\in J} a_{ij}$$

(b) Fubini: If $(a_{i,j})_{i,j\in I\times J}$ is absolutely summable, i.e. $\sum_{(i,j)\in I\times J} |a_{i,j}| < \infty$, then

$$\sum_{j\in J}\sum_{i\in I} a_{ij} = \sum_{(i,j)\in I\times J} a_{ij} = \sum_{i\in I}\sum_{j\in J} a_{ij}.$$

We'll first prove Fubini–Tonelli for $\mathcal{M}\otimes\mathcal{N}$-measurable functions, and deduce the $\mu\times\nu$-measurable version in HW.

Def. For a function $f : X\times Y \to Z$,

- for $x\in X$, call $f_x : Y \to Z$, given by $y \mapsto f(x,y)$, the *vertical fiber/section* of f over x / at x.
- for $y \in Y$, call $f^y : X \to Z$, given by $x \mapsto f(x,y)$, the *horizontal fiber/section* of f at y.

Similarly, for $R \subseteq X\times Y$, & $x_0 \in X$, $y_0\in Y$, call:

$$R_{x_0} := \{ y\in Y : (x_0, y) \in R\}$$

$$R_{y_0} := \{ x\in X : (x, y_0) \in R\}$$

the *vertical* and *horizontal fibers* of R at x_0 & y_0, resp.

Lemma. Let $(X,\mathcal{M})$ and $(Y,\mathcal{N})$ be measurable spaces.

(a) If $R \in \mathcal{M}\otimes\mathcal{N}$, then R_x and R^y are resp. in $\mathcal{N}$ and $\mathcal{M}$, for all $x\in X$ and $y \in Y$.

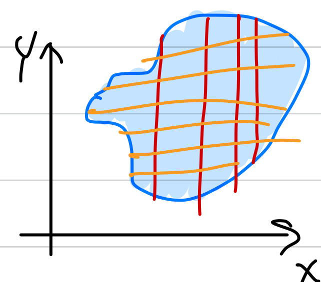
(b) If $f: X\times Y \to Z$ $(\mathcal{M}\otimes\mathcal{N}, \mathcal{L})$-meas., where $(Z,\mathcal{L})$ is a measurable space, then f_x and f^y are $(\mathcal{N},\mathcal{L})$ and $(\mathcal{M},\mathcal{L})$-meas. for all $x\in X$, $y\in Y$.

Proof. (a) Let $\mathcal{S}$ be the collection of all $R\in\mathcal{M}\otimes\mathcal{N}$ satisfying the conclusion $\forall_{x\in X}$ and $\forall_{y\in Y}$. Then $\mathcal{S}$ contains all rectangles $A\times B$, $A\in\mathcal{M}$ and $B\in\mathcal{N}$. Moreover, $\mathcal{S}$ is closed under ctbl unions and complements because $\left(\bigcup_n R_n\right)_x = \bigcup_n (R_n)_x$ and $(R^c)_x = (R_x)^c$, i.e. union and complement commute with fiber. So $\mathcal{S}$ is a σ-alg. containing all rectangles, hence $\mathcal{S} = \mathcal{M}\otimes\mathcal{N}$.

(b) This follows from (a) because preimage commutes with fiber: let $B\in\mathcal{L}$, then $f_x^{-1}(B) = (f^{-1}(B))_x \in \mathcal{N}$ by (a). $\square$

Fubini-Tonelli for sets. Let $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ be σ-finite measure spaces and let $R\in\mathcal{M}\otimes\mathcal{N}$. Then

$g: x \mapsto \nu(R_x)$, $X\to[0,\infty]$ and $h: y\mapsto \mu(R^y)$, $Y\to[0,\infty]$ are $\mathcal{M}$ and $\mathcal{N}$ measurable, and:



$$(*)\qquad \int \underbrace{\nu(R_x)}_{g(x)}\, d\mu(x) = \mu\times\nu(R) = \int \underbrace{\mu(R^y)}_{h(y)}\, d\nu(y).$$

Proof. Let $\mathcal{S}$ be the set of all $R\in\mathcal{M}\otimes\mathcal{N}$ s.t. the conclusion holds. Then $\mathcal{S}$ contains all rectangles $R=A\times B$, simply because $g(x) = \nu(R_x) = \nu(B)\cdot \mathbb{1}_A(x)$, i.e. $g = \nu(B)\cdot\mathbb{1}_A$, and $(*)$ by the def. of $\mu\times\nu$. First assume that μ,ν are finite.

Attempt 1. It's not hard to show that $\mathcal{S}$ is closed under complements. However, ctbl unions are difficult. If we could prove finite unions, then we would get ctbl unions from ctbl increasing unions, which are easy to deal with using MCT. Finite unions are still hard because if R_0 and R_1 overlap then $\mathbb{1}_{R_0 \cup R_1} \neq \mathbb{1}_{R_0} + \mathbb{1}_{R_1}$, so can't just appeal to the linearity of integral.

Attempt 2. Note that $\mathcal{S}$ actually contains the finite unions of rectangles because they are equal to finite disjoint unions of rectangles, so the indicator function is just the finite sum of indicators of rectangles, hence finite additivity of measures and linearity of integrals saves the day. Thus, $\mathcal{S}$ contains the alg. $\mathcal{A}$ generated by rectangles.

Monotone Class Lemma. If $\mathcal{S}$ contains an algebra $\mathcal{A}$ and is closed under $\bigcup\limits_n^{\uparrow}$ and $\bigcap\limits_n^{\downarrow}$, then $\mathcal{S} \supseteq \langle \mathcal{A} \rangle_\sigma$.

By this lemma, it's enough to show that $\mathcal{S}$ is closed under $\bigcup\limits_n^{\uparrow}$ and $\bigcap\limits_n^{\downarrow}$.

$\bigcup\limits_n^{\uparrow}$: If $R_n \in \mathcal{S}$ and $R := \bigcup\limits_n^{\uparrow} R_n$, then $h(y) := \nu(R^y) = \mu(\bigcup^{\uparrow} R_n^y) = \lim\limits_n \mu(R_n^y) = \lim\limits_n h_n(y)$, so $h = \lim\limits_n h_n$ hence is $\mathcal{N}$-measurable because each h_n is. $(*)$ for R follows from MCT.

$\bigcap\limits_n^{\downarrow}$: If $R_n \in \mathcal{S}$ and $R := \bigcap\limits_n^{\downarrow} R_n$, then $h(y) := \mu(R^y) = \mu(\bigcap\limits_n^{\downarrow} R_n^y) = (\text{by finiteness of } \mu) = \lim\limits_n \mu(R_n^y)$, so $h = \lim\limits_n h_n$ $\mathcal{N}$-measurable. $(*)$ holds by the DCT, by the finiteness of ν.

For σ-finite μ, ν, let $X = \Uparrow_n X_n$, where $X_n \in \mathcal{M}$ and $\mu(X_n) < \infty$, $Y_n \in \mathcal{N}$ and $\nu(Y_n) < \infty$. Then $X \times Y = \Uparrow_n X_n \times Y_n$ and given $R \in \mathcal{M} \otimes \mathcal{N}$, the statements hold for each $R_n := R \cap (X_n \times Y_n)$, so they hold for $R = \Uparrow_n R_n$ by upward monotonicity of measures and the MCT. $\square$

Call a collection $\mathcal{S} \subseteq \mathcal{P}(X)$ a monotone class if it is closed under $\Uparrow_n$ and $\Downarrow_n$.

Example. The set of boxes in $\mathbb{R}^d$ is a monotone class, but not a σ-algebra because it is not closed under complements or ctbl (even finite) unions. However, the monotone class generated by the finite unions of boxes is a σ-algebra as the following lemma asserts.

Monotone Class Lemma. Let $\mathcal{S}$ be the monotone class generated by an algebra $\mathcal{A}$. Then $\mathcal{S} = \langle \mathcal{A} \rangle_\sigma$.

Proof. It is enough to prove that $\mathcal{S}$ is an algebra because then every ctbl union $\bigcup_n B_n = \Uparrow_n (\bigcup_{i < n} B_i)$.

Closedness under complement. Let $\mathcal{C} := \{B \in \mathcal{S} : B^c \in \mathcal{S}\}$. $\mathcal{C} \supseteq \mathcal{A}$ and $\mathcal{C}$ is a monotone class: $(\Uparrow_n B_n)^c = \Downarrow_n B_n^c$ and $(\Downarrow_n B_n)^c = \Uparrow_n B_n^c$.

Closedness under finite unions. For each $B \in \mathcal{S}$, let

$$\mathcal{C}(B) := \{C \in \mathcal{S} : B \cup C \in \mathcal{S}\}.$$

This is a monotone class: $B \cup (\Uparrow_n C_n) = \Uparrow_n (B \cup C_n)$ and $B \cup (\Downarrow_n C_n) = \Downarrow_n (B \cup C_n)$.

If $A \in \mathcal{A}$, then $\mathcal{C}(A) \supseteq \mathcal{A}$. Hence $\mathcal{C}(A) = \mathcal{S}$. Thus, for each $B \in \mathcal{S}$, $\mathcal{C}(B) \supseteq \mathcal{A}$, so $\mathcal{C}(B)$ is the monotone class generated by $\mathcal{A}$, i.e. $\mathcal{C}(B) = \mathcal{S}$. Therefore, for all $B, C \in \mathcal{S}$, the union $B \cup C$ is also in $\mathcal{S}$. $\square$

Fubini-Tonelli Theorem. Let $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ be σ-finite measure spaces and let $f: X\times Y \to \overline{\mathbb{R}} := [-\infty,\infty]$ be an $\mathcal{M}\otimes\mathcal{N}$-measurable function.

(a) Tonelli. If $f\geq 0$, then the functions $g: x\mapsto \int f_x \, d\nu$ and $h: y\mapsto \int f^y \, d\mu$ are respectively $\mathcal{M}$ and $\mathcal{N}$ measurable and

$$\iint f(x,y)\, d\nu(y)\, d\mu(x) = \int f \, d\mu\times\nu = \iint f(x,y)\, d\mu(x)\, d\nu(y). \qquad (*)$$

(b) Fubini. If f is $\mu\times\nu$-integrable, then the functions g and h (as above) are μ and ν integrable and $(*)$ holds.

Proof. (a) This holds simple by linearity and Fubini-Tonelli for sets. For arbitrary $\mathcal{M}\otimes\mathcal{N}$-measurable $f\geq 0$, write f as an increasing limit of simple functions (s_n) and apply the fact that limits of measurable functions are measurable and MCT for $(*)$.

(b) Part (a) also shows that if $f\geq 0$ is also integrable, then $g(x)$ and $h(y)$ are finite for μ-a.e. $x\in X$, ν-a.e. $y\in Y$, by $(*)$ applied to f. Now for any $f\in L^1(X\times Y, \mathcal{M}\otimes\mathcal{N})$, $f=f^+-f^-$, and part (a) holds for f^+, f^-, so linearity gives the desired conclusion for f. $\square$

Remark. All conditions in this theorem are necessary; examples will be given in HW.

Remark. Usually, one first applies Tonelli to $|f|$ to show that f is integrable, and afterwards apply Fubini to f.